Problem 9.32  A two-element array consisting of two isotropic antennas separated by a distance $d$ along the $z$-axis is placed in a coordinate system whose $z$-axis points eastward and whose $x$-axis points toward the zenith. If $a_0$ and $a_1$ are the amplitudes of the excitations of the antennas at $z = 0$ and at $z = d$, respectively, and if $\delta$ is the phase of the excitation of the antenna at $z = d$ relative to that of the other antenna, find the array factor and plot the pattern in the $x$–$z$ plane for the following:

(a) $a_0 = a_1 = 1$, $\delta = \pi/4$, and $d = \lambda/2$
(b) $a_0 = 1$, $a_1 = 2$, $\delta = 0$, and $d = \lambda$
(c) $a_0 = a_1 = 1$, $\delta = -\pi/2$, and $d = \lambda/2$
(d) $a_0 = 1$, $a_1 = 2$, $\delta = \pi/4$, and $d = \lambda/2$
(e) $a_0 = 1$, $a_1 = 2$, $\delta = \pi/2$, and $d = \lambda/4$

Solution:
(a) Employing Eq. (9.110),

\[
F_a(\theta) = \left| \sum_{i=0}^{1} a_i e^{j\psi_i} e^{jkd \cos \theta} \right|^2 \\
= \left| 1 + e^{j((2\pi/\lambda)(\lambda/2) \cos \theta + \pi/4)} \right|^2 \\
= \left| 1 + e^{j(\pi \cos \theta + \pi/4)} \right|^2 = 4 \cos^2 \left[ \frac{\pi}{8} (4 \cos \theta + 1) \right].
\]

A plot of this array factor pattern is shown in Fig. 9-32(a).

**Figure P9.32:** (a) Array factor in the elevation plane for Problem 9.32(a).
(b) Employing Eq. (9.110),

\[
F_a(\theta) = \left| \sum_{i=0}^{1} a_i e^{j \Phi_i} e^{jk\theta i} \right|^2
\]

\[
= \left| 1 + 2e^{j((2\pi/\lambda)\lambda \cos \theta + 0)} \right|^2 = \left| 1 + 2e^{j2\pi \cos \theta} \right|^2 = 5 + 4 \cos(2\pi \cos \theta).
\]

A plot of this array factor pattern is shown in Fig. 9-32(b).

**Figure P9.32:** (b) Array factor in the elevation plane for Problem 9.32(b).
(c) Employing Eq. (9.110), and setting \( a_0 = a_1 = 1, \ \psi = 0, \ \psi_1 = \delta = -\pi/2 \) and \( d = \lambda/2 \), we have

\[
F_a(\theta) = \left| \sum_{i=0}^{1} a_i e^{j\psi_i} e^{jk_i \cos \theta} \right|^2 \\
= \left| 1 + e^{-j\pi/2} e^{j(2\pi/\lambda)(\lambda/2) \cos \theta} \right|^2 \\
= \left| 1 + e^{j(\pi \cos \theta - \pi/2)} \right|^2 \\
= 4 \cos^2 \left( \frac{\pi}{2} \cos \theta - \frac{\pi}{4} \right).
\]

A plot of the array factor is shown in Fig. 9-32(c).

![Figure P9.32: (c) Array factor in the elevation plane for Problem 9.32(c).](image-url)
(d) Employing Eq. (9.110), and setting \( a_0 = 1, \ a_1 = 2, \ \psi_0 = 0, \ \psi_1 = \delta = \pi/4, \) and \( d = \lambda/2, \) we have

\[
F_a(\theta) = \left| \sum_{i=0}^{1} a_i e^{j\psi_i} e^{jkd\cos\theta} \right|^2 \\
= \left| 1 + 2e^{j\pi/4} e^{j(2\pi/\lambda)(\lambda/2)\cos\theta} \right|^2 \\
= \left| 1 + 2e^{j(\pi\cos\theta + \pi/4)} \right|^2 \\
= 5 + 4\cos\left( \pi\cos\theta + \frac{\pi}{4} \right).
\]

A plot of the array factor is shown in Fig. 9-32(d).

\textbf{Figure P9.32:} (d) Array factor in the elevation plane for Problem 9.32(d).
(e) Employing Eq. (9.110), and setting $a_0 = 1$, $a_1 = 2$, $\psi_0 = 0$, $\psi_1 = \delta = \pi/2$, and $d = \lambda/4$, we have

$$F_a(\theta) = \left| \sum_{i=0}^{1} a_i e^{j\psi_i} e^{jkd \cos \theta} \right|^2$$

$$= \left| 1 + 2e^{j\pi/2} e^{j(2\pi/\lambda)(\lambda/4) \cos \theta} \right|^2$$

$$= \left| 1 + 2e^{j(\pi \cos \theta + \pi)/2} \right|^2$$

$$= 5 + 4 \cos \left( \frac{\pi}{2} \cos \theta + \frac{\pi}{2} \right) = 5 - 4 \sin \left( \frac{\pi}{2} \cos \theta \right).$$

A plot of the array factor is shown in Fig. 9-32(e).