Problem 4.24 Charge $Q_1$ is uniformly distributed over a thin spherical shell of radius $a$, and charge $Q_2$ is uniformly distributed over a second spherical shell of radius $b$, with $b > a$. Apply Gauss's law to find $E$ in the regions $R < a$, $a < R < b$, and $R > b$.

Solution: Using symmetry considerations, we know $D = \hat{R}D_R$. From Table 3.1, $ds = \hat{R}R^2\sin\theta\ d\theta\ d\phi$ for an element of a spherical surface. Using Gauss’s law in integral form (Eq. (4.29)),

$$\int_S D \cdot ds = Q_{\text{tot}},$$

where $Q_{\text{tot}}$ is the total charge enclosed in $S$. For a spherical surface of radius $R$,

$$\int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} (\hat{R}D_R) \cdot (\hat{R}R^2\sin\theta\ d\theta\ d\phi) = Q_{\text{tot}},$$

$$D_R R^2 (2\pi) [-\cos\theta]_{\theta=0}^{\pi} = Q_{\text{tot}},$$

$$D_R = \frac{Q_{\text{tot}}}{4\pi R^2}.$$

From Eq. (4.15), we know a linear, isotropic material has the constitutive relationship $D = \varepsilon E$. Thus, we find $E$ from $D$.

(a) In the region $R < a$,

$$Q_{\text{tot}} = 0, \quad E = \hat{R}E_R = \frac{\hat{R}Q_{\text{tot}}}{4\pi R^2 \varepsilon} = 0 \ (\text{V/m}).$$

(b) In the region $a < R < b$,

$$Q_{\text{tot}} = Q_1, \quad E = \hat{R}E_R = \frac{\hat{R}Q_1}{4\pi R^2 \varepsilon} \ (\text{V/m}).$$

(c) In the region $R > b$,

$$Q_{\text{tot}} = Q_1 + Q_2, \quad E = \hat{R}E_R = \frac{\hat{R}(Q_1 + Q_2)}{4\pi R^2 \varepsilon} \ (\text{V/m}).$$