Problem 5.37  Obtain an expression for the self-inductance per unit length for the parallel wire transmission line of Fig. 5-27(a) in terms of \(a\), \(d\), and \(\mu\), where \(a\) is the radius of the wires, \(d\) is the axis-to-axis distance between the wires, and \(\mu\) is the permeability of the medium in which they reside.

Solution:

Figure P5.37: Parallel wire transmission line.

Let us place the two wires in the \(x\)-\(z\) plane and orient the current in one of them to be along the \(+z\)-direction and the current in the other one to be along the \(-z\)-direction, as shown in Fig. P5.37. From Eq. (5.30), the magnetic field at point \(P = (x,0,z)\) due to wire 1 is

\[
B_1 = \hat{\phi} \frac{\mu I}{2\pi r} = \hat{y} \frac{\mu I}{2\pi x},
\]

where the permeability has been generalized from free space to any substance with permeability \(\mu\), and it has been recognized that in the \(x\)-\(z\) plane, \(\hat{\phi} = \hat{y}\) and \(r = x\) as long as \(x > 0\).
Given that the current in wire 2 is opposite that in wire 1, the magnetic field created by wire 2 at point \( P = (x, 0, z) \) is in the same direction as that created by wire 1, and it is given by

\[
B_2 = \hat{y} \frac{\mu I}{2\pi(d-x)}.
\]

Therefore, the total magnetic field in the region between the wires is

\[
B = B_1 + B_2 = \hat{y} \frac{\mu I}{2\pi} \left( \frac{1}{x} + \frac{1}{d-x} \right) = \hat{y} \frac{\mu Id}{2\pi x(d-x)}.
\]

From Eq. (5.91), the flux crossing the surface area between the wires over a length \( l \) of the wire structure is

\[
\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} = \int_{a}^{d-a} \int_{z=0}^{z=2\pi l} \left( \frac{\mu Id}{2\pi x(d-x)} \right) \cdot (\hat{y} \, dx \, dz)
\]

\[
= \frac{\mu Id}{2\pi} \left( \frac{1}{d} \ln \left( \frac{x}{d-x} \right) \right) \Bigg|_{x=a}^{x=d-a}
\]

\[
= \frac{\mu Id}{2\pi} \left( \ln \left( \frac{d-a}{a} \right) - \ln \left( \frac{a}{d-a} \right) \right)
\]

\[
= \frac{\mu Id}{2\pi} \times 2 \ln \left( \frac{d-a}{a} \right) = \frac{\mu Il}{\pi} \ln \left( \frac{d-a}{a} \right).
\]

Since the number of ‘turns’ in this structure is 1, Eq. (5.93) states that the flux linkage is the same as magnetic flux: \( \Lambda = \Phi \). Then Eq. (5.94) gives a total inductance over the length \( l \) as

\[
L = \frac{\Lambda}{I} = \frac{\Phi}{I} = \frac{\mu l}{\pi} \ln \left( \frac{d-a}{a} \right) \quad \text{(H)}.
\]

Therefore, the inductance per unit length is

\[
L' = \frac{L}{l} = \frac{\mu}{\pi} \ln \left( \frac{d-a}{a} \right) \approx \frac{\mu}{\pi} \ln \left( \frac{d}{a} \right) \quad \text{(H/m)},
\]

where the last approximation recognizes that the wires are thin compared to the separation distance (i.e., that \( d \gg a \)). This has been an implied condition from the beginning of this analysis, where the flux passing through the wires themselves have been ignored. This is the thin-wire limit in Table 2-1 for the two wire line.