Problem 4.32  A circular ring of charge of radius \(a\) lies in the \(x-y\) plane and is centered at the origin. Assume also that the ring is in air and carries a uniform density \(\rho_l\).

(a) Show that the electrical potential at \((0,0,z)\) is given by

\[
V = \frac{\rho_l a}{2\varepsilon_0 \left(a^2 + z^2\right)^{1/2}}.
\]

(b) Find the corresponding electric field \(E\).

Solution:

(a) For the ring of charge shown in Fig. P4.32, using Eq. (3.67) in Eq. (4.48c) gives

\[
V(R) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho_l}{R} dl' = \frac{1}{4\pi\varepsilon_0} \int_{0}^{2\pi} \frac{\rho_l}{\sqrt{a^2 + r^2 - 2ar \cos(\phi' - \phi) + z^2}} a d\phi'.
\]

Point \((0,0,z)\) in Cartesian coordinates corresponds to \((r,\phi,z) = (0,\phi,z)\) in cylindrical coordinates. Hence, for \(r = 0\),

\[
V(0,0,z) = \frac{1}{4\pi\varepsilon_0} \int_{\phi'=0}^{2\pi} \frac{\rho_l}{\sqrt{a^2 + z^2}} a d\phi' = \frac{\rho_l a}{2\varepsilon_0 \sqrt{a^2 + z^2}}.
\]

(b) From Eq. (4.51),

\[
E = -\nabla V = -\hat{z} \frac{\rho_l a}{2\varepsilon_0} \frac{\partial}{\partial z} \left(a^2 + z^2\right)^{-1/2} = \hat{z} \frac{\rho_l a}{2\varepsilon_0} \frac{z}{(a^2 + z^2)^{3/2}} \text{ (V/m)}.
\]