Problem 4.22  Given the electric flux density

\[ \mathbf{D} = \mathbf{\hat{x}} 2(x + y) + \mathbf{\hat{y}} (3x - 2y) \quad (\text{C/m}^2) \]

determine

(a) \( \rho_v \) by applying Eq. (4.26).

(b) The total charge \( Q \) enclosed in a cube 2 m on a side, located in the first octant with three of its sides coincident with the x-, y-, and z-axes and one of its corners at the origin.

(c) The total charge \( Q \) in the cube, obtained by applying Eq. (4.29).

Solution:

(a) By applying Eq. (4.26)

\[ \rho_v = \nabla \cdot \mathbf{D} = \frac{\partial}{\partial x} (2x + 2y) + \frac{\partial}{\partial y} (3x - 2y) = 0. \]

(b) Integrate the charge density over the volume as in Eq. (4.27):

\[ Q = \int_V \nabla \cdot \mathbf{D} \, dV = \int_{x=0}^{2} \int_{y=0}^{2} \int_{z=0}^{2} 0 \, dx \, dy \, dz = 0. \]

(c) Apply Gauss’ law to calculate the total charge from Eq. (4.29)

\[ Q = \int \mathbf{D} \cdot d\mathbf{s} = F_{\text{front}} + F_{\text{back}} + F_{\text{right}} + F_{\text{left}} + F_{\text{top}} + F_{\text{bottom}}, \]

\[ F_{\text{front}} = \int_{y=0}^{2} \int_{z=0}^{2} (\mathbf{\hat{x}} 2(x + y) + \mathbf{\hat{y}} (3x - 2y)) \bigg|_{x=2} \cdot (\mathbf{\hat{x}} \, dz \, dy) \]

\[ = \int_{y=0}^{2} \int_{z=0}^{2} 2(x + y) \bigg|_{x=2} \, dz \, dy = \left( 2z \left( 2y + \frac{1}{2} y^2 \right) \bigg|_{z=0}^{2} \right) = 24, \]

\[ F_{\text{back}} = \int_{y=0}^{2} \int_{z=0}^{2} (\mathbf{\hat{x}} 2(x + y) + \mathbf{\hat{y}} (3x - 2y)) \bigg|_{x=0} \cdot (-\mathbf{\hat{x}} \, dz \, dy) \]

\[ = -\int_{y=0}^{2} \int_{z=0}^{2} 2(x + y) \bigg|_{x=0} \, dz \, dy = -\left( zy^2 \bigg|_{z=0}^{2} \right) = -8, \]

\[ F_{\text{right}} = \int_{x=0}^{2} \int_{z=0}^{2} (\mathbf{\hat{x}} 2(x + y) + \mathbf{\hat{y}} (3x - 2y)) \bigg|_{y=2} \cdot (\mathbf{\hat{y}} \, dz \, dx) \]

\[ = \int_{x=0}^{2} \int_{z=0}^{2} (3x - 2y) \bigg|_{y=2} \, dz \, dx = \left( z \left( \frac{3}{2} x^2 - 4x \right) \bigg|_{z=0}^{2} \right) = -4, \]
Thus $Q = \int \mathbf{D} \cdot ds = 24 - 8 - 4 - 12 + 0 + 0 = 0$. 

\[
F_{\text{left}} = \int_{x=0}^{2} \int_{z=0}^{2} (\hat{x}2(x+y) + \hat{y}(3x-2y)) \left. \right|_{y=0}^{y=2} \cdot (-\hat{y} \, dz \, dx) \\
= - \int_{x=0}^{2} \int_{z=0}^{2} (3x-2y) \left. \right|_{y=0}^{y=2} \, dz \, dx = - \left( \left. z \left( \frac{3}{2}x^2 \right) \right|_{z=0}^{z=2} \right)^2 \left|_{x=0}^{x=2} \right. = -12,
\]

\[
F_{\text{top}} = \int_{x=0}^{2} \int_{z=0}^{2} (\hat{x}2(x+y) + \hat{y}(3x-2y)) \left. \right|_{z=2}^{z=0} \cdot (\hat{z} \, dy \, dx) \\
= \int_{x=0}^{2} \int_{z=0}^{2} 0 \left. \right|_{z=2}^{z=0} \, dy \, dx = 0,
\]

\[
F_{\text{bottom}} = \int_{x=0}^{2} \int_{z=0}^{2} (\hat{x}2(x+y) + \hat{y}(3x-2y)) \left. \right|_{z=0}^{z=2} \cdot (\hat{z} \, dy \, dx) \\
= \int_{x=0}^{2} \int_{z=0}^{2} 0 \left. \right|_{z=0}^{z=2} \, dy \, dx = 0.
\]