EEL 4473
Spectral Domain Techniques
and
Diffraction Theory
References:


*These notes are essentially based on reference 1 above.*
Elementary example of (two-dimensional) diffraction
**Huygens’ Principle**: Each point on a propagating wavefront can be considered as a secondary source radiating a spherical wave.

For the slit, an element of width $dx$ in the aperture radiates (a cylindrical wave in this 2-D case) into the medium to the right of the aperture.

For this infinitesimal element, assume that the strength of the secondary source is $E_o dx$.

The contribution of element $E_o dx$ to the field at point $P$ a distance $r'$ away will be of the form:
\[ dE_p = C E_o dx \frac{1}{\sqrt{r'}} e^{-jkr'} \]

\[ k = \omega \sqrt{\mu \varepsilon} \]

\[ C = \text{const.} \]

**For** \( r \gg dx \)

\[ \frac{1}{\sqrt{r}} \approx \frac{1}{\sqrt{r'}} , \quad e^{-jkr'} \approx e^{-jk(r-x\sin\theta)} \]
\[ E_P = \int_{-a/2}^{a/2} dE_P = \frac{CE_o}{\sqrt{r}} e^{-jkr} \int_{-a/2}^{a/2} e^{-jkx \sin \theta} \, dx \]

\[ = \frac{CE_o}{\sqrt{r}} a e^{-jkr} \frac{\sin \left( \frac{\pi a \sin \theta}{\lambda} \right)}{\pi a \sin \theta} \frac{\lambda}{\lambda}, \quad \lambda = \frac{2\pi}{k} \]
Huygens’ Principle gives the field at point P as the superposition of cylindrical waves from all parts of the aperture and produces the interference pattern shown.

\[ \Theta := -\frac{\pi}{2}, -\frac{\pi}{2} + 0.0001 \ldots \frac{\pi}{2} \]

\[ E_p(\theta, \alpha') := \begin{cases} 1 & \text{if } \theta = 0 \\ \frac{\sin(\pi \cdot \alpha' \cdot \sin(\theta))}{(\pi \cdot \alpha' \cdot \sin(\theta))} & \text{otherwise} \end{cases} \quad a' := \frac{a}{\lambda} \]
Note that the amplitude of the radiated field increases with aperture width, $a$, while the angular width of the main beam reduces.

As the aperture width is increased, the gain of the antenna is increased and its beamwidth is reduced.

\[ E_p(\theta, a) = \begin{cases} 1 & \text{if } \theta = 0 \\ \frac{\sin(\pi \cdot a' \cdot \sin(\theta))}{(\pi \cdot a' \cdot \sin(\theta))} & \text{otherwise} \end{cases} \]
Diffraction Theory Using Plane Waves

Though there has been much study in diffraction theory, the underlying principle of spherical waves radiated by known fields – Huygens’ Principle – has remained.

Though spherical waves are a “natural” physical entity they are mathematically cumbersome. Planes waves on the other hand are a simple mathematical entity but are not physical.

We know however that physical fields can be represented by the superposition, be it discrete or continuous, of plane waves traveling in different direction.

This superposition is known as the \textit{Angular Spectrum}\footnote{Angular Spectrum}
Diffraction Theory Using Plane Waves

The angular plane wave spectrum concept is a mathematically simple approach to develop diffraction theory.

It is also a fundamentally more precise approach compared with other approaches as approximations are made later in the analysis rather than earlier. In fact, the analysis contains the near field information as well as the far.
Diffraction Theory Using Plane Waves

Recall that for temporal signals:

\[ f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \]

Possibly finite in extent (physical)
Infinite in extent (non-physical)

We also know:

\[ F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \]
Elementary example of (two-dimensional) diffraction

Typical plane wave of the angular spectrum

Incident plane wave
Diffraction Theory Using Plane Waves

Suppose, instead of the cylindrical waves of the last example we represent the field diffracted by a slit as a superposition of plane waves one of which is shown.

The angle $\theta$ describes the directions of the component plane waves. It will be more convenient to use $\sin \theta$ rather than $\theta$ as the angular variable, however.

$$A(\theta) = F(\sin \theta) = F(s)$$
Diffraction Theory Using Plane Waves

If the set of plane waves is described by the spectrum function $F(\sin \theta)$, the contribution of this single plane wave of elemental amplitude $F(\sin\theta) \, d(\sin\theta)$, to the field point P is:

$$dE_p(x, z) = F(\sin \theta) d(\sin \theta) e^{-jk(x\sin\theta + z\cos\theta)}$$

Then:

$$E_p(x, z) = \int_{-\infty}^{\infty} F(\sin \theta) e^{-jk(x\sin\theta + z\cos\theta)} d(\sin \theta)$$

For now assume the range of integration is artificially extended beyond its natural limits of ±1 for analytical convenience. We’ll later see that waves traveling in directions such that $|\sin\theta| > 1$ is far from artificial.
Define the field over the aperture as

\[ f(x) = E_P(x, 0) \]

Then

\[ E_P(x, 0) = f(x) = \int_{-\infty}^{\infty} F(\sin \theta) e^{-jkx \sin \theta} d(\sin \theta) \]

This is a Fourier integral and can be inverted to give the angular spectrum in terms of the aperture field as

\[ F(\sin \theta) = \frac{1}{\lambda} \int_{-\infty}^{\infty} f(x) e^{jkx \sin \theta} dx, \quad k = \frac{2\pi}{\lambda} \]

(see next page for details)
Read details for homework.

\[ f(x) = \int_{-\infty}^{\infty} F(\sin \theta) e^{-j k x \sin \theta} d(\sin \theta) \]

\[ f(x) e^{j k x \sin \theta} = e^{j k x \sin \theta} \int_{-\infty}^{\infty} F(\sin \theta) e^{-j k x \sin \theta} d(\sin \theta) \]

\[ \int_{-\infty}^{\infty} f(x) e^{j k x \sin \theta} \, dx = \int_{-\infty}^{\infty} \left[ e^{j k x \sin \theta} \int_{-\infty}^{\infty} F(u) e^{-j k u} \, du \right] \, dx \]

\[ \int_{-\infty}^{\infty} f(x) e^{j k x \sin \theta} \, dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u) e^{j k x \sin \theta} e^{-j k u} \, du \, dx \]

\[ \int_{-\infty}^{\infty} f(x) e^{j k x \sin \theta} \, dx = \int_{-\infty}^{\infty} F(u) \int_{-\infty}^{\infty} e^{j k x \sin \theta} e^{-j k u} \, dx \, du \]
\[ \int_{-\infty}^{\infty} f(x) e^{jkx \sin \theta} \, dx = \int_{-\infty}^{\infty} F(u) \int_{-\infty}^{\infty} e^{jkx \sin \theta} e^{-jkxu} \, dx \, du \]

Recall: \[ \delta(t - t_o) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\pm j\omega(t-t_o)} \, d\omega \]

\[ \int_{-\infty}^{\infty} f(x) e^{jkx \sin \theta} \, dx = \int_{-\infty}^{\infty} F(u) \frac{1}{k} \int_{-\infty}^{\infty} e^{-j(kx)(u-\sin \theta)} \, d(kx) \, du \]

\[ = 2\pi \delta(u-\sin \theta) \]

\[ = \frac{2\pi}{k} \int_{-\infty}^{\infty} F(u) \delta(u-\sin \theta) \, du = \frac{2\pi}{k} F(\sin \theta) \]

\[ \Rightarrow F(\sin \theta) = \frac{k}{2\pi} \int_{-\infty}^{\infty} f(x) e^{jkx \sin \theta} \, dx = \frac{1}{\lambda} \int_{-\infty}^{\infty} f(x) e^{jkx \sin \theta} \, dx \]

Which is the desired result.
For the slit example: \( f(x) = \begin{cases} E_o & |x| \leq \frac{a}{2} \\ 0 & \text{otherwise} \end{cases} \)

thus

\[
F(\sin \theta) = \frac{1}{\lambda} \int_{-\infty}^{\infty} f(x) e^{jkx \sin \theta} \, dx = \frac{1}{2\lambda} \int_{-\frac{a}{2}}^{\frac{a}{2}} E_o e^{jkx \sin \theta} \, dx = E_o \frac{a}{\lambda} \frac{\sin \left( \frac{\pi a \sin \theta}{\lambda} \right)}{\pi a \sin \theta}
\]

Which has the same angular dependence as the field derived from Huygens’ principle. Recall:

\[
E_p = \frac{CE_o}{\sqrt{r}} a e^{-jkr} \frac{\sin \left( \frac{\pi a \sin \theta}{\lambda} \right)}{\pi a \sin \theta}
\]
This *example* demonstrates a result that we will later show is generally true; that the angular spectrum gives the far-field of the radiation.

Furthermore, the angular spectrum is simply the Fourier transform of the field across the radiating aperture.

The result can be extended to three dimensions and the reciprocity theorem to be established allows the technique to be used for receiving as well as transmitting apertures.

The results apply to both the near-field and far-field of the aperture, and so this plane wave approach is used extensively in near-field measurements of fields as well as for antennas.
Plane Wave Representation of Two-Dimensional Fields

Fields are assumed uniform in the $y$-direction, i.e., the fields are 2-dimensional.

Direction of Plane Wave

$$P(x, z)$$

$$\vec{r} = x\hat{a}_x + z\hat{a}_z$$

$$\hat{a} = \sin \theta \hat{a}_x + \cos \theta \hat{a}_z$$

let: $$s = \sin \theta, \quad c = \cos \theta$$

$$\Rightarrow \hat{a} = s\hat{a}_x + c\hat{a}_z$$

$$c = \sqrt{1 - s^2}$$

$s$ and $c$ are the direction cosines of the plane.
If the electric field at the origin is $\vec{E}_0$ then the electric field at $P$ is:

$$\vec{E}(x,z) = \vec{E}_0 e^{-jk\hat{a}\cdot\vec{r}}$$

$$= \vec{E}_0 e^{-jk(sx+cz)}$$

It will be convenient to resolve the field into two orthogonal linearly polarized plane waves, one with the electric field vector parallel to the $x$-$z$ plane and the other with the electric field vector perpendicular to the plane.

The first has its magnetic field vector pointing entirely in the transverse ($y$) direction and is called the transverse magnetic TM case.

The second has its electric field pointing entirely in the transverse ($y$) direction and is referred to as the transverse electric (TE) case.
Angular Spectrum for TM Fields

\[ \mathbf{E}(x, z) = \mathbf{E}_o e^{-jk\mathbf{a} \cdot \mathbf{r}} \]

\[ = \mathbf{E}_o e^{-jk(sx+cz)} \]

\[ E_x(x, z) = E_o \cos \theta e^{-jk(sx+cz)} \]

\[ E_z(x, z) = -E_o \sin \theta e^{-jk(sx+cz)} \]

\[ E_y = 0 \]

\[ H_y(x, z) = \frac{E_o}{\eta} e^{-jk(sx+cz)} \]

\[ H_x = H_z = 0 \]
A set of plane waves traveling in different directions is represented by the spectrum function \( A(\theta) \), such that the electric field amplitude of the plane wave traveling in the direction \( \theta \) is \( A(\theta) \, d\theta \). The angular spectrum can be continuous, discrete, or a mixture of the two.

We replace \( E_o \) by \( A(\theta) \, d\theta \) and integrate over the range of angles for which the angular spectrum is defined. For example:

\[
E_x(x, z) = \int_{-\infty}^{\infty} A(\theta) \cos \theta e^{-j k (sx + cz)} \, d\theta
\]

It will be convenient to express the angular spectrum in terms of \( s = \sin \theta \) rather than \( \theta \).
\[ s = \sin \theta, \quad ds = \cos \theta d\theta = c d\theta, \quad c = \sqrt{1 - s^2} \]

Let \( A(\theta) = F(s) \)

Then: \( E_x(x,z) = \int_{-\infty}^{\infty} A(\theta) \cos \theta e^{-j k (sx+cz)} \, d\theta = \int_{-\infty}^{\infty} F(s) \cos \theta e^{-j k (sx+cz)} \frac{1}{\cos \theta} \, ds \)

\[ = \int_{-\infty}^{\infty} F(s) e^{-j k (sx+cz)} \, ds \]

\[ E_z(x,z) = -E_o \sin \theta e^{-j k (sx+cz)} \]

\[ E_z(x,z) = - \int_{-\infty}^{\infty} A(\theta) \sin \theta e^{-j k (sx+cz)} \, d\theta = - \int_{-\infty}^{\infty} F(s) \frac{S}{c} e^{-j k (sx+cz)} \, ds \]

\[ H_y(x,z) = \frac{E_o}{\eta} e^{-j k (sx+cz)} \]

\[ H_y(x,z) = \frac{1}{\eta} \int_{-\infty}^{\infty} A(\theta) e^{-j k (sx+cz)} \, d\theta = \frac{1}{\eta} \int_{-\infty}^{\infty} F(s) \frac{1}{c} e^{-j k (sx+cz)} \, ds \]

\[ E_y = 0, \quad H_x = H_z = 0 \]

Read details for homework.
\[ E_x(x, z) = \int_{-\infty}^{\infty} F(s) e^{-jk(sx+cz)} \, ds \]

\[ E_z(x, z) = -\int_{-\infty}^{\infty} F(s) \frac{s}{c} e^{-jk(sx+cz)} \, ds \]

\[ H_y(x, z) = \frac{1}{\eta} \int_{-\infty}^{\infty} F(s) \frac{1}{c} e^{-jk(sx+cz)} \, ds \]

\[ E_y = 0, \quad H_x = H_z = 0 \]

Note that all the field components for the TM wave have been expressed in terms of one spectrum function \( F(s) \)
Evanescent Waves

Consider: \[ dE_x(x,z) = F(s)ds e^{-jk(sx+cz)} \]

Again, this is the contribution of that plane wave to the angular spectrum of amplitude \( F(s)ds \) traveling in the direction making the angle \( \theta = \sin^{-1}s \) to the \( z \)-axis.

When \( |s| \leq 1 \), \( \theta \) lies in the range \(-\pi/2 \leq \theta \leq \pi/2\) and the elemental plane wave is the usual homogeneous type with which we’re familiar. The plane wave travels with the speed of light in the medium and transfers power to \( z \geq 0 \).
Evanescent Waves

When \( |s| \geq 1 \), (assuming that \( s \) is real) the character of the wave changes since

\[
c = \sqrt{1 - s^2} = \pm j\chi
\]

where \( \chi \) is real and positive.

Then

\[
dE_x(x,z) = F(s) ds e^{-jk_s x} e^{-jkc z} = F(s) ds e^{-jk_s x} e^{-k\chi z}
\]

Where we have chosen the minus sign \(-j\chi = c\) so that the fields remain finite as \( z \to +\infty \)

This is an \textit{inhomogeneous plane wave} in that the amplitude is no longer constant over planes of constant phase. Inhomogeneous plane waves have the following three properties:
**Evanescent Waves**

a) The direction of propagation of the wave is along the $x$-axis, parallel to the aperture plane, whether in the positive or negative $x$-direction depending upon the sign of $s$.

b) The amplitude of the field decreases exponentially in the $+z$-direction, away from the aperture plane, hence they are **evanescent** in nature (evanescent means disappearing).

For all but the smallest values of $\chi$ the evanescent waves become negligibly small at distances more than a few wavelengths from the aperture. The distance from the aperture plane at which the amplitude has decreased to a fraction $e^{-1}$ of its value at $z = 0$ is

$$z = \frac{\lambda}{2\pi\chi}$$
Evanescent Waves

c) The phase velocity is obtained via

\[ e^{-jksx} e^{j\omega t} \]

and is

\[ v_p = \frac{1}{s \sqrt{\mu \varepsilon}} \]

is slower than that of a homogeneous plane wave since \(|s| > 1\).

These waves do carry power, but it is not propagated in the region \(z > 0\). It merely travels back and forth across the aperture and can be thought of as being stored there. Because they are closely “bound” to the surface (the aperture plane) these waves are known as electromagnetic surface waves.
Analogy to Total Internal Reflection

\[ \theta_i > \theta_{\text{critical}} \]

- Dielectric
- Air
- Incident plane wave
- Aperture plane
Analogies to Total Internal Reflection

For $\theta_i > \theta_{\text{critical}}$, the incident plane wave is totally reflected, but the fields in the air region are not zero, since boundary conditions require that tangential $E$ and $H$ be continuous at the boundary. The fields on the air side of the boundary are evanescent waves which are local to the boundary and carry no energy across it.

Recall Snell’s law which related the angle of incidence $\theta_i$ and the angle of transmission $\theta_t$:

$$s = \sin \theta_i = \sqrt{\varepsilon} \sin \theta_t$$

Clearly when $\theta_i > \theta_{\text{critical}} = \sin^{-1}(\varepsilon^{-1/2})$ then $s > 1$ but remains real. But these are precisely the conditions we obtained for evanescent waves.
We see that a complete representation of the 2-dimensional fields in $z > 0$ in terms of an angular spectrum of plane waves $F(s)$, $s = \sin \theta$, must include spectral components over the entire range $-\infty < s < \infty$.

If $|s| \leq 1$, the plane wave propagates freely as a homogeneous plane wave into $z > 0$.

This is known as the visible or propagating portion of the angular spectrum.
If $|s| > 1$, the plane waves are inhomogeneous and do not propagate into the medium but only store reactive power in the aperture plane, and are evanescent in that they are negligible at more than a few wavelengths from the aperture plane.

This is the **invisible**, **non-propagating**, or **reactive** part of the angular spectrum.
Aperture Field: The Fourier Transform of the Angular Spectrum

Consider the $x$-component of the electric field over the aperture plane $z = 0$:

$$E_{ax}(x) = E_x(x, 0)$$

For TM polarization,

$$E_{ax}(x) = \int_{-\infty}^{\infty} F(s) e^{-jksx} ds$$

The aperture field $E_a(x)$ is the Fourier transform of the angular plane wave spectrum $F(s)$.

Applying the inversion formula,

$$F(s) = \frac{1}{\lambda} \int_{-\infty}^{\infty} E_{ax}(x) e^{+jksx\sin\theta} dx$$
Aperture Field: The Fourier Transform of the Angular Spectrum

This means that if we know the tangential component of the electric field over the aperture (at \( z = 0 \)) we can deduce the angular spectrum in terms of which we know the fields everywhere in the region \( z > 0 \).

\[
E_{ax}(x) \leftrightarrow F(s)
\]

This statement is exact, and it applies to TM waves in 2-dimensions.

Soon we’ll extend this result to TE waves as well as to 3-dimensions.
Return to the slit example

Incident plane wave (wholly $x$-directed)

$$E_{x\text{--inc}}(x,z) = E_o e^{-j kz}$$

The diffracted field will be of the TM type

$\frac{a}{2}$

Boundary conditions require that the tangential field be zero on the metal screen, thus:

$$E_{ax}(x) = \begin{cases} E_o & |x| < \frac{a}{2} \\ 0 & \text{otherwise} \end{cases}$$

$0$

$-\frac{a}{2}$

Perfectly Conducting Slit
Return to the slit example

\[ F(s) = \frac{1}{\lambda} \int_{-\infty}^{\infty} E_{ax}(x) e^{+j k x \sin \theta} \, dx = \frac{E_o}{\lambda} \int_{-\frac{a}{2}}^{\frac{a}{2}} e^{+j k x \sin \theta} \, dx = E_o \frac{a}{\lambda} \sin \left( \frac{\pi a s}{\lambda} \right) \]

\[ = E_o \frac{a}{\lambda} \text{sinc} \left( \frac{a s}{\lambda} \right), \quad \text{sinc}(u) = \frac{\sin \pi u}{\pi u} \]

\[ F(s)_{\text{max}} = F(0) = E_o \frac{a}{\lambda} \]

\[ \text{as} \frac{a}{\lambda} \to \infty, \quad E_o \lim_{\frac{a}{\lambda} \to \infty} \frac{a}{\lambda} \text{sinc} \left( \frac{a s}{\lambda} \right) = E_o \delta(s) \quad \text{This is a plane wave traveling in the } \theta = 0 \text{ direction as it should.} \]
Return to the slit example

For \( \frac{a}{\lambda} \) finite:

\[ F(s, a') := \begin{cases} 1 & \text{if } s = 0 \\ \frac{a'}{\sin(\pi a' s)} & \text{otherwise} \end{cases} \]

\( s = -5, -4.99, \ldots, 5 \)

Amplitude of spectrum small in evanescent region

Spectrum essentially constant over the propagating range of angles and continues at approximately the same level into the evanescent region.
The example suggests that the angular spectrum and diffraction pattern are equivalent.

In fact, at large distances from the aperture, the angular dependence of the fields is approximately that of the angular spectrum, and the approximation improves the larger the distance from the aperture.
Far-Field Approximated by Angular Spectrum

Any 2-dimensional TM field propagating into the region $z \geq 0$ can be represented by the single spectrum function $F(s)$.

As we found for the $x$-component:

$$E_x(x, z) = \int_{-\infty}^{\infty} F(s) e^{-jk(sx+cz)} ds$$  

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This determines $E_x$ everywhere in $z \geq 0$ by a single integral though it may be difficult to evaluate.

There is one important general result that we can obtain immediately for the field at very large distances from a diffracting aperture of finite size.
Direction of Plane Wave

\[ P : (x, z) = (r \sin \psi, r \cos \psi) \]
Substitute: \((x, z) = (r \sin \psi, r \cos \psi)\)

Into:

\[
E_x (x, z) = \int_{-\infty}^{\infty} F(s) e^{-jk(sx+cz)} ds
\]

\[
= \int_{-\infty}^{\infty} F(s) e^{-jkr(s \sin \psi + c \cos \psi)} ds
\]

\[
= \int_{-\infty}^{\infty} F(\sin \theta) e^{-jkr(\sin \theta \sin \psi + \cos \theta \cos \psi)} d(\sin \theta)
\]

\[
= \int_{-\infty}^{\infty} F(\sin \theta) e^{-jkr \cos(\theta - \psi)} d(\sin \theta)
\]
Now suppose that $P$ is many wavelengths away from the origin, i.e.,

$$kr \gg 1$$

For most range of $\psi$ evanescent waves will not contribute.

As $\theta$ varies over the range of real angles the phase in the exponential

$$e^{-jkr \cos(\theta-\psi)}$$

will rotate rapidly through many multiples of $2\pi$ except when $\cos(\psi - \theta)$ is “stationary”.

The stationary phase condition occurs when $\theta = \psi$, i.e. for that wave in the angular spectrum traveling in the direction $(0, P)$
Next suppose that $F(s)$ is a bounded and continuous function of $s$, i.e., it does not contain any discrete delta function components. This means that the aperture field can be non-zero only over a finite area – a very practical condition.

Then as $\theta$ varies over the integration range, neighboring values of $F(s)$ have approximately the same amplitude but appear in the integrand with opposite phases and almost cancel each other.

This happens everywhere except in the direction in which the stationary phase condition is satisfied.

The only non-negligible contribution to the integrand will come from the direction $\theta = \psi$ and its immediate neighborhood.
\[ f(x) = \cos(K \cdot g(x)) \]
\[ g(x) = \cos \theta \]
\[ \frac{dg}{dx} = 0 \text{ for } \theta = \frac{\pi}{3} \]
We will find that the field at point \( P \) is

\[
E_x(r, \psi) = CF (\sin \psi) e^{-jkr}, \quad kr \to \infty
\]

where \( C \) is a constant to be determined.
The Principle of Stationary Phase  Read details for homework.

Used to evaluate integrals of the form:

\[ I = \int_{a}^{b} f(x)e^{jKg(x)} \, dx \]

where \( f(x) \) and \( g(x) \) are real, continuous, and of bounded variation and \( K \) is a \textit{large, positive, real} number.

Suppose that \( g(x) \) is stationary at only one point \( x = x_o, a < x_o < b \), i.e.,

\[ g'(x_o) = 0, \quad a < x_o < b \]
The Method of Stationary Phase  

Read details for homework.

Based on the previous argument, the only significant contribution to the integral $I$ will be in the neighborhood of $x_o$.

Thus expand $g(x)$ in a Taylor series about $x = x_o$,

$$g(x) = g(x_o) + \left(\frac{g'}{x_o}\right)^0 (x-x_o) + \frac{1}{2} g''(x_o)(x-x_o)^2 + \ldots$$

and assume that $f(x)$ is slowly varying. Then

$$I = \int_a^b f(x) e^{jKg(x)} \, dx \approx \int_a^b f(x_o) e^{jK\left[g(x_o)+\frac{1}{2}g''(x_o)(x-x_o)^2\right]} \, dx$$

$$= f(x_o) e^{jKg(x_o)} \int_a^b e^{\frac{jK}{2}g''(x_o)(x-x_o)^2} \, dx$$
The Method of Stationary Phase  Read details for homework.

Since by assumption there is only one singular point, we may extend the limits of integration to infinity.

\[ I \approx f(x_o) e^{jKg(x_o)} \int_{-\infty}^{\infty} e^{j\frac{1}{2}Kg''(x_o)(x-x_o)^2} \, dx \]

Let \( \xi^2 = \frac{1}{2} Kg''(x_o)(x-x_o)^2 \), \( 2\xi \, d\xi = Kg''(x_o)(x-x_o) \, dx \)

\[ \Rightarrow dx = \frac{2\xi}{Kg''(x_o)(x-x_o)} \, d\xi = \frac{2\sqrt{\frac{1}{2}Kg''(x_o)(x-x_o)^2}}{Kg''(x_o)(x-x_o)} \, d\xi \]

\[ = \frac{\sqrt{2}}{\sqrt{Kg''(x_o)}} \, d\xi \]

assume that \( g''(x_o) > 0 \)
The Method of Stationary Phase

\[
dx = \frac{\sqrt{2}}{\sqrt{Kg''(x_o)}} \, d\xi
\]

\[
\Rightarrow I \approx f(x_o) e^{jKg(x_o)} \int_{-\infty}^{\infty} e^{j \frac{1}{2} Kg''(x_o) (x-x_o)^2} \, dx
\]

\[
= f(x_o) e^{jKg(x_o)} \sqrt{\frac{2}{Kg''(x_o)}} \int_{-\infty}^{\infty} e^{j\xi^2} \, d\xi
\]

We have the well-known result that (see A. Papoulis, *The Fourier Integral*, McGraw-Hill, 1962, pp. 141, 220):

\[
\int_{-\infty}^{\infty} e^{j\xi^2} \, d\xi = \sqrt{j\pi}
\]
The Method of Stationary Phase

\[ I \approx f(x_o)e^{jKg(x_o)} \sqrt{\frac{2}{Kg''(x_o)}} \int_{-\infty}^{\infty} e^{j\xi^2} d\xi = f(x_o)e^{jKg(x_o)} \sqrt{\frac{j2\pi}{Kg''(x_o)}} \]

\[ = f(x_o)e^{j\left[Kg(x_o) + \frac{\pi}{4}\right]} \sqrt{\frac{2\pi}{Kg''(x_o)}} \]
The Method of Stationary Phase  

Read details for homework.

if \( g''(x_o) < 0, \)

\[
g(x) = g(x_o) + \left. \frac{g'(x_o)}{g''(x_o)} \right|_{x=x_o} (x-x_o) - \frac{1}{2} \left| g''(x_o) \right| (x-x_o)^2 + \ldots
\]
\[
I \approx f(x_o) e^{iKg(x_o)} \int_{-\infty}^{\infty} e^{-\frac{1}{2}K|g''(x_o)|(x-x_o)^2} \, dx
\]

\[
\xi^2 = -\frac{1}{2}K|g''(x_o)|(x-x_o)^2, \quad 2\xi d\xi = -K|g''(x_o)|(x-x_o) \, dx
\]

\[
\Rightarrow dx = -\frac{2\xi}{K|g''(x_o)|(x-x_o)} \, d\xi = -\frac{2j\sqrt{\frac{1}{2}K|g''(x_o)|(x-x_o)^2}}{K|g''(x_o)|(x-x_o)} \, d\xi
\]

\[
= -\frac{j\sqrt{2}}{\sqrt{K|g''(x_o)|}} \, d\xi, \quad g''(x_o) < 0
\]
\[ I \approx f(x_o) e^{jKg(x_o)} \int_{-\infty}^{\infty} e^{-j \frac{1}{2} K|g''(x_o)|(x-x_o)^2} \, dx \]

\[ = -f(x_o) e^{jKg(x_o)} j \sqrt{\frac{2}{K|g''(x_o)|}} \int_{-\infty}^{\infty} e^{j\xi^2} \, d\omega \]

\[ = -f(x_o) e^{jKg(x_o)} j \sqrt{\frac{j2\pi}{K|g''(x_o)|}} \]

\[ = f(x_o) e^{j \left[ Kg(x_o) + \pi + \frac{\pi}{2} + \frac{\pi}{4} \right]} \sqrt{\frac{2\pi}{K|g''(x_o)|}} \]

\[ = f(x_o) e^{j \left[ Kg(x_o) + \frac{4\pi}{4} + \frac{2\pi}{4} + \frac{\pi}{4} \right]} \sqrt{\frac{2\pi}{K|g''(x_o)|}} \]

\[ = f(x_o) e^{j \left( \frac{7\pi}{4} - \frac{\pi}{4} \right)} \sqrt{\frac{2\pi}{K|g''(x_o)|}}, \quad g''(x_o) < 0 \]

Read details for homework.
The Method of Stationary Phase  

Read details for homework.

\[
I = \lim_{K \to \infty} \int_{-\infty}^{\infty} e^{\frac{1}{2} K g''(x_o) \xi^2} \, dx \to f(x_o) \sqrt{\frac{2\pi}{K}} \frac{1}{|g''(x_o)|} e^{j \left[ K g(x_o) + \frac{\pi}{4} \text{sgn}[g''(x_o)] \right]}
\]

\[
\text{sgn}(u) = \begin{cases} 
+1 & u > 0 \\
-1 & u < 0 
\end{cases}
\]

This is the stationary phase result for the asymptotic evaluation, as \( K \) tends to infinity, of single integrals of the type considered.
Applying the stationary phase to the diffraction integral, identify:

\[
I = \lim_{K \to \infty} \int_{-\infty}^{\infty} e^{\frac{j}{2}Kg''(x_o)\xi^2} \, dx \to f(x_o) \sqrt{\frac{2\pi}{K \, |g''(x_o)|}} e^{j \left[ Kg(x_o) + \frac{\pi}{4} \text{sgn}[g''(x_o)] \right]}
\]

\[
E_x(x, z) = \int_{-\infty}^{\infty} F(\sin \theta) e^{-jkr \cos(\theta - \psi)} \, d(\sin \theta)
\]

\[
x \leftrightarrow \theta
\]

\[
x_o \leftrightarrow \psi, \text{ the stationary point, since } \frac{d}{d\theta} \cos(\theta - \psi) = 0 \text{ at } \theta = \psi
\]

\[
K \leftrightarrow kr
\]

\[
f(x) \leftrightarrow \cos \theta F(\sin \theta)
\]

\[
g(x) \leftrightarrow -\cos(\theta - \psi)
\]
\[ E_x (x, z) = \lim_{kr \to \infty} \int_{-\infty}^{\infty} F(\sin \theta) e^{-jkr\cos(\theta-\psi)} d(\sin \theta) \]

\[ \rightarrow \cos \psi F(\sin \psi) \frac{\sqrt{2\pi}}{kr} \frac{1}{|g''(x_o)|} e^{j \left[-kr - \frac{\pi}{4} \text{sgn}[g''(x_o)]\right]} \]

\[ g(\theta) = -\cos(\theta - \psi), \quad g'(\theta) = \sin(\theta - \psi), \quad g''(\theta) = \cos(\theta - \psi) \]

\[ \Rightarrow g''(\psi) = \cos(\psi - \psi) = 1 > 0 \]

\[ E_x (x, z) \rightarrow \cos \psi F(\sin \psi) \sqrt{\frac{2\pi}{kr}} e^{-jkr} e^{-j\frac{\pi}{4}} = \cos \psi F(\sin \psi) \sqrt{\frac{j2\pi}{kr}} e^{-jkr} \]

\[ = \cos \psi F(\sin \psi) \sqrt{\frac{j\lambda}{r}} e^{-jkr} \]

Which specifies the constant \( C \) included earlier.
The remaining field components are found in a similar way:

\[ E_x(x, z) = \int_{-\infty}^{\infty} F(s) e^{-jk(sx + cz)} ds \]

\[ E_z(x, z) = -\int_{-\infty}^{\infty} F(s) \frac{S}{c} e^{-jk(sx + cz)} ds \]

\[ H_y(x, z) = \frac{1}{\eta} \int_{-\infty}^{\infty} F(s) \frac{1}{c} e^{-jk(sx + cz)} ds \]

\[ E_y = 0, \quad H_x = H_z = 0 \]

\[ E_x(x, z) \rightarrow \cos \psi F(\sin \psi) \sqrt{\frac{j\lambda}{r}} e^{-jkr} \]

\[ E_z(x, z) \rightarrow -\sin \psi F(\sin \psi) \sqrt{\frac{j\lambda}{r}} e^{-jkr} \]

\[ H_y(x, z) \rightarrow \frac{1}{\eta} F(\sin \psi) \sqrt{\frac{j\lambda}{r}} e^{-jkr} \]
More rigorous methods, such as the method of steepest descent confirm this result and in fact shows that the result obtained is the first term of an asymptotic series in ascending odd powers of $K^{-1/2}$, hence our result is indeed a correct asymptotic result.

Further details can be found in:

For the sake of economy of symbols, let us rewrite the results replacing $\psi$ by $\theta$.

\[
E_x(x, z) \rightarrow \cos \theta F(\sin \theta) \sqrt{\frac{j\lambda}{r}} e^{-jkr}
\]

\[
E_z(x, z) \rightarrow -\sin \theta F(\sin \theta) \sqrt{\frac{j\lambda}{r}} e^{-jkr}
\]

\[
H_y(x, z) \rightarrow \frac{1}{\eta} F(\sin \theta) \sqrt{\frac{j\lambda}{r}} e^{-jkr}
\]
\[ E_\theta (r, \theta) = F (\sin \theta) \sqrt{\frac{j \lambda}{r}} e^{-jkr} = \eta H_y (r, \theta) \]
We see that the far field for a TM polarized wave radiating in two dimensions is a cylindrical wave (asymptotically as $kr \rightarrow \infty$). Note that in this case the axis of the cylinder is the $y$-axis.

Note also that the local character of the field is (asymptotically) a plane wave.

Also, Poynting’s theorem applied to $E_\theta$ and $H_y$ shows that the power flow in the far field is radial at all points with magnitude:

$$S_r(r, \theta) = \frac{\lambda}{2\eta r} |F(\sin \theta)|^2 \left[ \text{Watts/meter-unit width} \right]$$

Called the Angular Power Spectrum.
The Antenna Far-Field Theorem

a) The far field of an antenna is asymptotically equal to its angular spectrum.

b) The angular spectrum for an antenna is the Fourier transform of its aperture field, which is the tangential component of the field in the aperture plane of the antenna.

We proved this result for one of the orthogonally polarized components of a 2 – dimensional field.

The theorem is also true for the 3 – dimensional case.
Though the result is asymptotically true, i.e. for

\[ kr \rightarrow \infty \]

the Rayleigh condition that

\[ r \geq \frac{2a^2}{\lambda} \]

is sufficient, where \( a \) is the largest dimension of the aperture.
We have formulated our results in terms of the tangential components of the electric field. It is of course perfectly valid to use the tangential magnetic components, though this is not as common. The theorem holds as worded in either case.

The Antenna Far Field Theorem is more commonly known in a more abbreviated form: *The far field of an antenna is given asymptotically by the Fourier transform of its aperture field.*

We break it into two statements, a) and b), to emphasize the fact that the Fourier transform relation between the angular spectrum and the aperture field is *exact*, with the approximation entering only in the *evaluation* of the far field.
Fraunhofer Diffraction

In optics, the far field pattern is referred to as the *Fraunhofer* diffraction pattern, with the only minor difference being that the fields are specified over a plane rather than over a sphere (circle).
If the extent of the Fraunhofer plane is small compared with its distance \( r_o \) from the aperture plane, then

\[
\sin \theta \approx \frac{x}{r_o}
\]

and from our earlier result,

\[
E_\theta (r, \theta) = \eta H_y (r, \theta) = F (\sin \theta) \sqrt{\frac{j\lambda}{r}} e^{-jkr}
\]

\[
\approx F \left( \frac{x}{r_o} \right) \sqrt{\frac{j\lambda}{r_o}} e^{-jkr} \approx F \left( \frac{x}{r_o} \right) \sqrt{\frac{j\lambda}{r_o}} e^{-jk_r \left( 1 + \frac{x^2}{2r_o^2} \right)}
\]

using \( r = \sqrt{z^2 + x^2} \)

\[
= \sqrt{r_o^2 + x^2} = r_o \sqrt{1 + \frac{x^2}{r_o^2}} \approx r_o \left( 1 + \frac{x^2}{2r_o^2} \right)
\]
in the exponential
Complete 2-Dimensional Fields (TM Polarization)

Slide 24:

\[
\begin{bmatrix}
E_x(x, z) \\
E_z(x, z) \\
H_y(x, z)
\end{bmatrix} = \int_{-\infty}^{\infty} F_{TM}^{TM}(s) \begin{bmatrix}
1 \\
-\frac{s}{c} \\
\frac{1}{\eta c}
\end{bmatrix} e^{-j(k(sx+cz))} ds
\]

Which has the asymptotic solution for \( kr \to \infty \)

\[
\begin{bmatrix}
E_x(x, z) \\
E_z(x, z) \\
H_y(x, z)
\end{bmatrix} \to \begin{bmatrix}
\cos \theta \\
-\sin \theta \\
\frac{1}{\eta}
\end{bmatrix} F_{TM}^{TM}(\sin \theta) \sqrt{\frac{j\lambda}{r}} e^{-jkr}
\]
Complete 2-Dimensional Fields (TM Polarization)

Or in polar form:

\[ E_{\theta}(r, \theta) = \eta H_{y}(r, \theta) \rightarrow \sqrt{\frac{\lambda}{r}} F_{TM}(\sin \theta) e^{-j\left(\frac{kr-\pi}{4}\right)} \]
The TE polarized waves (which give the remaining field components) can be constructed in exactly the same way in terms of a second angular spectrum $F_{TE}(s)$, or they can be readily deduced by the principle of duality.
**Principle of Duality**

Suppose the fields $E_1$ and $H_1$ satisfy Maxwell’s equations:

\[
\nabla \times \tilde{E}_1 = -j\omega \mu \tilde{H}_1 \\
\nabla \times \tilde{H}_1 = j\omega \varepsilon \tilde{E}_1
\]

Make the substitution

\[
\tilde{E}_2 = \eta \tilde{H}_1 \\
\tilde{H}_2 = -\frac{1}{\eta} \tilde{E}_1
\]

then

\[
-\nabla \times \eta \tilde{H}_2 = -j\omega \mu \frac{1}{\eta} \tilde{E}_2 \\
\nabla \times \frac{1}{\eta} \tilde{E}_2 = -j\omega \varepsilon \eta \tilde{H}_2
\]
Principle of Duality

\[-\nabla \times \eta \tilde{H}_2 = -j\omega \mu \frac{1}{\eta} \tilde{E}_2\]

\[\nabla \times \tilde{H}_2 = \frac{j\omega \mu}{\eta^2} \tilde{E}_2\]

\[\nabla \times \frac{1}{\eta} \tilde{E}_2 = -j\omega \epsilon \eta \tilde{H}_2\]

\[\nabla \times \tilde{E}_2 = -j\omega \epsilon \eta^2 \tilde{H}_2\]

\[\nabla \times \tilde{H}_2 = \frac{j\omega \mu}{\mu} \tilde{E}_2 = j\omega \epsilon \tilde{E}_2\]

\[\nabla \times \tilde{E}_2 = -j\omega \epsilon \frac{\mu}{\epsilon} \tilde{H}_2 = -j\omega \mu \tilde{H}_2\]
Principle of Duality

\[ \nabla \times \vec{E}_1 = -j\omega \mu \vec{H}_1 \rightarrow \vec{E}_2 = \eta \vec{H}_1 \rightarrow \nabla \times \vec{H}_2 = j\omega \varepsilon \vec{E}_2 \]

\[ \nabla \times \vec{H}_1 = j\omega \varepsilon \vec{E}_1 \rightarrow \vec{H}_2 = -\eta^{-1} \vec{E}_1 \rightarrow \nabla \times \vec{E}_2 = -j\omega \mu \vec{H}_2 \]

This shows that \( E_2 \) and \( H_2 \) are also solutions of Maxwell’s equations as applied to sinusoidal fields in a lossless, source-free media.

This means that if one solution for the fields, \( E_1, H_1 \), has been found, then a second solution, \( E_2, H_2 \), can be obtained as above.

It is important to note that different boundary conditions must apply in order that the two solutions be truly independent.
\[
\begin{bmatrix}
E_x(x,z) \\
E_z(x,z) \\
H_y(x,z)
\end{bmatrix} = \int_{-\infty}^{\infty} F_{TM}(s) \begin{bmatrix}
1 \\
-\frac{s}{c} \\
\frac{1}{\eta c}
\end{bmatrix} e^{-jk(sx+cz)} ds
\]

\[
\begin{bmatrix}
\bar{E}_z = \eta \bar{H}_1 \\
\bar{H}_2 = -\eta^{-1} \bar{E}_1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\eta H_x(x,z) \\
\eta H_z(x,z) \\
\eta^{-1} E_y(x,z)
\end{bmatrix} = \int_{-\infty}^{\infty} F_{TM}(s) \begin{bmatrix}
1 \\
-\frac{s}{c} \\
\frac{1}{\eta c}
\end{bmatrix} e^{-jk(sx+cz)} ds
\[
\begin{bmatrix}
H_x(x,z) \\
H_z(x,z) \\
E_y(x,z)
\end{bmatrix} = \int_{-\infty}^{\infty} F_{TM}(s) \begin{bmatrix}
\frac{1}{\eta} \\
\frac{1}{\eta} \\
\frac{1}{c}
\end{bmatrix} e^{-jk(sx+cz)} ds
\]

Note the order of the terms

\[
\begin{bmatrix}
E_y(x,z) \\
H_x(x,z) \\
H_z(x,z)
\end{bmatrix} = \int_{-\infty}^{\infty} \frac{1}{c} \begin{bmatrix}
1 \\
1 \\
F_{TE}(s)
\end{bmatrix} \begin{bmatrix}
\frac{1}{\eta} \\
\frac{s}{\eta}
\end{bmatrix} e^{-jk(sx+cz)} ds
\]

Factor out the \( c \) so that \( F_{TE} \) is the Fourier transform of the aperture field \( E_y \) consistent with \( F_{TM} \) as the Fourier transform of \( E_x \).
\[
\begin{bmatrix}
E_x(x, z) \\
E_z(x, z) \\
H_y(x, z)
\end{bmatrix} = \int_{-\infty}^{\infty} F_{TM}(s) \begin{bmatrix}
1 \\
-\frac{s}{c} \\
\frac{1}{\eta c}
\end{bmatrix} e^{-j(k(sx + cz))} ds
\]

\[
\begin{bmatrix}
E_y(x, z) \\
H_x(x, z) \\
H_z(x, z)
\end{bmatrix} = \int_{-\infty}^{\infty} F_{TE}(s) \begin{bmatrix}
1 \\
\frac{1}{\eta} \\
-\frac{s}{\eta}
\end{bmatrix} e^{-j(k(sx + cz))} ds
\]

Aperture Plane

\[
\vec{E}
\]

\[
E_{ax} E_{ay}
\]
\[ E_y(x, z) = \int_{-\infty}^{\infty} F_{TE}(s) e^{-jk(sx + cz)} ds \]

\[ E_{ay}(x) = E_y(x, 0) = \int_{-\infty}^{\infty} F_{TE}(s) e^{-jksx} ds \]

\[ \Rightarrow F_{TE}(s) = \frac{1}{\lambda} \int_{-\infty}^{\infty} E_{ay}(x) e^{jksx} dx \quad \text{TE Angular Spectrum} \]

Fourier transform of \( E_{ay}(x) \)
The far-field solution is again obtained by applying the duality principle to

\[ E_{\theta}(r, \theta) = \eta H_{y}(r, \theta) \rightarrow \sqrt{\frac{\lambda}{r}} F_{TM}(\sin \theta) e^{-j\left(kr - \frac{\pi}{4}\right)} \]

or

\[ \vec{E}_2 = \eta \vec{H}_1, \quad \vec{H}_2 = -\frac{1}{\eta} \vec{E}_1 \]

\[ \eta H_{\theta}(r, \theta) = -\eta \frac{E_{y}(r, \theta)}{\eta} \]

\[ \Rightarrow E_{y}(r, \theta) = -\eta H_{\theta}(r, \theta) \rightarrow \sqrt{\frac{\lambda}{r}} F_{TM}(\sin \theta) e^{-j\left(kr - \frac{\pi}{4}\right)} \]

\[ \Rightarrow E_{y}(r, \theta) = -\eta H_{\theta}(r, \theta) = \sqrt{\frac{\lambda}{r}} \cos \theta F_{TE}(\sin \theta) e^{-j\left(kr - \frac{\pi}{4}\right)} \]

\[ \uparrow \text{remember this:} \quad \frac{1}{c} F_{TM}(s) \frac{1}{F_{TE}(s)} \]
Which is the cylindrical wave shown below.

\[ E_y(r, \theta) = -\eta H_\theta(r, \theta) = \sqrt{\frac{\lambda}{r}} \cos \theta F_{TE}(\sin \theta) e^{-j\left(kr - \frac{\pi}{4}\right)} \]
In summary, the complete electromagnetic field (in two dimensions) radiated into \( z \geq 0 \) is given in terms of two independent angular spectra,

\[
\begin{bmatrix}
  F_{TM}(s) \\
  F_{TE}(s)
\end{bmatrix}
= \frac{1}{\lambda} \int_{-\infty}^{\infty} \begin{bmatrix}
  E_{ax}(x) \\
  E_{ay}(x)
\end{bmatrix} e^{+jk_{sx}} dx
\]

with far field asymptotically given for \( kr \to \infty \) as

\[
\vec{E}(r, \theta) = \hat{a}_\theta E_\theta(r, \theta) + \hat{a}_y E_y(r, \theta)
\]

\[
= \sqrt{\frac{j\lambda}{r}} \left[ \hat{a}_\theta F_{TM}(\sin \theta) + \hat{a}_y \cos \theta F_{TE}(\sin \theta) \right] e^{-jkr}
\]

and

\[
\vec{H}(r, \theta) = \frac{1}{\eta} \hat{a}_r \times \vec{E}(r, \theta)
\]
The fields represented by the two spectra are not only complete, they are unique.

**Uniqueness Theorem of Electromagnetics**

Consider a source-free volume $V$ completely enclosed by surface $A$. Suppose there are two solutions, $(E_1, H_1)$ and $(E_2, H_2)$ which both satisfy Maxwell's equations within $V$ and on $A$. Then by superposition the difference fields $(E_1 - E_2, H_1 - H_2)$ will also satisfy Maxwell's equations.

Now apply the divergence theorem with $\hat{n}$ the outward normal to the surface …
\[-\oint_A (\vec{E}_1 - \vec{E}_2) \times (\vec{H}_1 - \vec{H}_2) \cdot \hat{n} da\]

\[= \int_V \left[ \frac{\partial}{\partial t} \left( \frac{\mu}{2} |\vec{H}_1 - \vec{H}_2|^2 + \frac{\varepsilon}{2} |\vec{E}_1 - \vec{E}_2|^2 \right) + \sigma |\vec{E}_1 - \vec{E}_2|^2 \right] dv\]

The normal components of the difference fields \(E_1 - E_2\) or \(H_1 - H_2\) do not enter into the integral on the left-hand-side (see next page). Hence if either tangential \(E\) or tangential \(H\) is specified uniquely at all points on \(A\), then the integral in the left side is zero.
\[(\vec{A} \times \vec{B}) \cdot \hat{n} = \left( A_n \hat{n} + \vec{A}_t \right) \times \left( B_n \hat{n} + \vec{B}_t \right) \cdot \hat{n} \]

\[
= \left[ A_n \hat{n} \times B_n \hat{n} = 0 + \left( A_n \hat{n} \right) \times \vec{B}_t + \vec{A}_t \times \left( B_n \hat{n} \right) + \vec{A}_t \times \vec{B}_t \right] \cdot \hat{n}
\]

\[
= \left( A_n \hat{n} \right) \times \vec{B}_t \cdot \hat{n} = 0 + \vec{A}_t \times \left( B_n \hat{n} \right) \cdot \hat{n} = 0 + \left( \vec{A}_t \times \vec{B}_t \right) \cdot \hat{n}
\]

\[
= \left( \vec{A}_t \times \vec{B}_t \right) \cdot \hat{n}
\]
But for the right-hand-side to be zero, both

\[
\left( \vec{E}_1 - \vec{E}_2 \right) \equiv 0 \quad \text{and} \quad \left( \vec{H}_1 - \vec{H}_2 \right) \equiv 0
\]

must be identically zero throughout the entire volume.

Hence there is one and only one solution throughout \( V \). That is, the solution is unique within \( V \) if either \( E_{\text{tangential}} \) or \( H_{\text{tangential}} \) is specified over the entire surface \( A \).
The uniqueness theorem applied to the source-free region such as the half cylinder $z \geq 0$ defined by the aperture plane and the circle of radius $r$. 

Aperture $A$ 

Aperture Plane $z = 0$ 

$E_{ax}(x) \leftrightarrow F_{TM}(s)$ 

$E_{ay}(x) \leftrightarrow F_{TE}(s)$ 

Spectral Domain Techniques and Diffraction Theory - 2-D Fields
Over the aperture plane the tangential electric field must be completely specified before the angular spectrum $F_{TE}$ and $F_{TM}$ can be determined.

For large $r$ the field over the curved surface of the cylinder will have the far-field forms derived previously.

These far field components are everywhere tangential to the surface, and as $r$ tends to infinity the amplitudes of all field components tend to zero.

Hence the 2-D fields in the half-space $z \geq 0$ represented by the angular spectrum are unique.